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GENERAL PELEG TYPE KKM THEOREM WITH APPLICATIONS

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Abstract

In our previous work[7], we discuss the continuous selection theorem on pseudo space. The main object of this paper is to introduce the general Peleg's KKM theorem on pseudo spaces and establish some new fixed point results. As applications, we derive some new existence results for system of variational inequalities.

Key words**:** Pseudo spaces, q-convex set, KKM theorem, Peleg's theorem, Upper (Lower) inverse, Fixed point theorem, System of variational inequalities, System of minimax inequalities.

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1. Preliminaries

In very recent years, Marchi and Martínez-Legaz [8] extended the Peleg's theorem [9] to H-spaces and obtain some generalizations of Fan-Browder's fixed point theorem, also discussed the Ky Fan type inequalities and the intersection theorem for set with convex sections. Park and Kim [11] gave a Peleg's theorem on G-convex spaces and applications to a whole intersection theorem and to prove existence theorems of equilibrium points in qualitative games. In our recent works [6], we discuss the general Peleg type KKM theorem on pseudo H-spaces and establish some new fixed point results. In this paper, we first introduce the general Peleg type KKM theorem on pseudo spaces and establish some new fixed point results. We also derive some new coincidence theorems and the geometric formulation of the Ky Fan type minimax inequality. As an application, we discuss the system of variational inequalities and Ky Fan type minimax inequalities..

Let X be a non-empty set. We denote by 2^X the family of all subsets of X, by $|X|$ the cardinality of X. If X is a subset of a vector space, $co(X)$ denotes the convex hull of X. Let Δ^n denote the standard n-simplex $co\{e_1,\dots,e_{n+1}\}$, where e_i is the i th unit vector in R^{n+1} for $i=1,2,\dots,n+1$.

We first recall the definition the pseudo space as follows.

Definition 1.1. [7] A triple $(X,D,\{q_A\})$ is said to be a *pseudo spaces* if X is a topological space, D be a nonempty set and for each nonempty finite subset A of D, there is a corresponding mapping $q_A:\Delta^{|A|-1}\to P(X)$ is an upper semi-continuous mapping with nonempty compact values such that the following two conditions hold: (1) there is an upper semi-continuous mapping $q_B:\Delta^{|A|-1}\to P(X)$ with nonempty compact values such that q_B is a restriction of q_A on $\Delta^{|B|-1}$ for all $\varnothing\neq B\subset A$ and (2) there is an upper semi-continuous mapping $q_C:\Delta^{|C|-1}\to P(X)$ with nonempty compact values such that q_A is a restriction of q_C on $\Delta^{|A|-1}$ for all $A\subset C\subset D$. If $D=X$, the triple $(X,D,\{q_A\})$

can be written by $(X,\{q_A\})$.

If the mapping q_A is single-valued and we set $\Gamma(A)=q_A(\Delta^{|A|-1})$ for each nonempty finite subset A of X, then (X,D,Γ) form a G-convex space (One can refer to [11]). Example of pseudo space we can find in our recent results [7].

If A, B are two nonempty finite subsets with $A\subset B$ and $|A|+1=|B|$. Then $\Delta^{|A|-1}$ is a face of $\Delta^{|B|-1}$ corresponding to A, the set $\Delta^{|A|-1}$ shall homeomorphic to the set $\Delta^{|A|-1}\times\{0\}\subset\Delta^{|B|-1}$. In the case like this, we shall replace the notation "$q_A(\Delta^{|A|-1}\times\{0\})\subset q_B(\Delta^{|B|-1})$" by "$q_A(\Delta^{|A|-1})\subset q_B(\Delta^{|B|-1})$". The other cases (i.e., $|A|+1<|B|$) are similar to the before one in the sequel.

Let P and Q be two non-empty sets in a pseudo space $(X,D,\{q_A\})$, we say that P is *q-convex relative to* Q if for each nonempty finite subset A of D with $A\subset Q$, we have $q_A(\Delta^{|A|-1})\subset P$. We note that if Q is non-empty and P is q-convex relative to Q, then P is automatically non-empty. If $P=Q$, we say P is a *q-convex set* of X.

The *lower inverse* of $F:X\to 2^Y$ is the set-valued map $F^-:Y\to 2^X$ defined by $F^-(B)=\{x\in X: B\cap F(x)\neq\emptyset\}$ for $B\in 2^Y$. The *upper inverse* of $F:X\to 2^Y$ is the set-valued map $F^+:Y\to 2^X$ defined by $F^+(B)=\{x\in X: B\subset F(x)\}$ for $B\in 2^Y$. A subset W of Y is called *compactly closed* (*compactly open*, resp.) if, for any compact set K in Y, $W\cap K$ is closed (open, resp.) in K.

For a pseudo space $(X,D,\{q_A\})$, a real-valued function $f:X\to R$ is called *q-quasi-convex* (resp., *q-quasi-concave*) if the sets $f^{-1}((-\infty,\lambda])$ (resp., $f^{-1}([\lambda,+\infty))$) are q-convex for all $\lambda\in R$.

For each $n\in N$, we say that the family $\{\Omega_k\}_{k=1}^n$ has *intersection property* with respect to F if the following property hold:

$$F^-\Big(\bigcap_{k=1}^n\Omega_k\Big)=\bigcap_{k=1}^n F^-(\Omega_k).$$

We say that the family $\{\Omega_k\}_{k=1}^n$ has *intersection property* with respect to F on nonempty set K if the family $\{\Omega_k\cap K\}_{k=1}^n$ has *intersection property* with respect to F, that is, the following property hold:

$$F^{-}(\bigcap_{k=1}^{n}\Omega_k \cap K) = \bigcap_{k=1}^{n} F^{-}(\Omega_k \cap K).$$

We note that if F is a single valued or vector valued function, then it satisfy the above both intersection properties.

Through out this paper, we denote $I = \{1, 2, ..., m\}$, $X = \prod_{k\in I} X_k$ and $q_A = \prod_{k\in I} q_{A_k}$ whenever X_k and q_{A_k}, $k \in I$, are given.

2. Main Results

We first need the generalized KKM theorem due to Peleg [9].

Peleg Theorem: For $k \in I$, let N_k be a nonempty finite set and $\Delta^{|N_k|-1}$ be $(|N_k|-1)$-simplex in $R^{|N_k|}$. If C_i^k, $i \in \{1, 2, ..., |N_k|\}$ and $k \in I$, are closed subsets of $\Delta^{|N_1|-1} \times \Delta^{|N_2|-1} \times \cdots \times \Delta^{|N_m|-1}$ such that for each $Q_k \subset N_k$, $k \in I$,

$$\Delta^{|N_1|-1} \times \Delta^{|N_2|-1} \times \cdots \times \Delta^{|Q_k|-1} \times \cdots \times \Delta^{|N_m|-1} \subset \bigcup_{j=1}^{|Q_k|} C_j^k,$$

where $\Delta^{|Q_k|-1}$ denotes the face of $\Delta^{|N_k|-1}$ corresponding to Q_k. Then we have

$$\bigcap_{k=1}^{m}\bigcap_{i=1}^{|N_k|} C_i^k \neq \emptyset.$$

Now we shall use the Peleg theorem to prove our main result.

Theorem 2.1: *Let* $(X_k, D_k, \{q_{A_k}\})$ *be a pseudo space,* N_k *be any nonempty finite subset of* D_k *for* $k \in I$. *If* C_x^k, $x \in N_k$ *and* $k \in I$, *are compactly closed subsets of* $X = \prod_{k\in I} X_k$ *that has intersection property with respect to* q *on any nonempty compact set in* X *such that for each* $A_k \in 2^{N_k}$, $\{\emptyset\}$, $k \in I$,

$$q_{N_1}(\Delta^{|A_1|-1}) \times q_{N_2}(\Delta^{|A_2|-1}) \times \cdots \times q_{N_m}(\Delta^{|A_m|-1}) \subset \bigcap_{k=1}^{m} \bigcup_{x \in A_k} C_x^k,$$

then

$$\bigcap_{k=1}^{m} \bigcap_{x \in N_k} C_x^k \neq \varnothing.$$

Furthermore, if the product space X *is compact or some finite intersection of* C_x^k *is compact, then*

$$\bigcap_{k=1}^{m} \bigcap_{x \in D_k} C_x^k \neq \varnothing.$$

Proof: Let $q_N : \Delta^{|N_1|-1} \times \cdots \times \Delta^{|N_m|-1} \to 2^X$ be the mapping defined by

$$q_N(\alpha^1, \alpha^2, \ldots, \alpha^m) = (q_{N_1}(\alpha^1), q_{N_2}(\alpha^2), \ldots, q_{N_m}(\alpha^m)),$$

where $\alpha^k \in \Delta^{|N_k|-1}, k \in I$, $N = N_1 \times N_2 \times \ldots N_m$. Then we can deduce that

$$\Delta^{|A_1|-1} \times \cdots \times \Delta^{|A_m|-1} \subset q_N^-(\bigcap_{k=1}^{m} \bigcup_{x \in A_k} E_x^k) \subset \bigcap_{k=1}^{m} \bigcup_{x \in A_k} q_N^-(E_x^k).$$

By Peleg Theorem and the intersection property,

$$q_N^-(\bigcap_{k=1}^{m} \bigcap_{x \in N_k} E_x^k) = \bigcap_{k=1}^{m} \bigcap_{x \in N_k} q_N^-(E_x^k) \neq \varnothing.$$

Therefore,

$$\bigcap_{k=1}^{m} \bigcap_{x \in N_k} C_x^k \supset \bigcap_{k=1}^{m} \bigcap_{x \in N_k} E_x^k \neq \varnothing.$$

Furthermore, if the product space X is compact or there are $k \in I$ and $x \in N_k$ such that C_x^k is compact or some finite intersection of C_x^k is compact, then

$$\bigcap_{k=1}^{m} \bigcap_{x \in D_k} C_x^k \neq \varnothing. \qquad \blacksquare$$

Now we have the following corollary which is slight generalized the results derived by Marchi and Martínez-Legaz ([8], Lemma 3).

Corollary 2.2: *For* $k \in I$*, let* X_k *be a topological space,* N_k *any nonempty finite set of* D_k *and* $\{\Gamma_A^k\}_{\emptyset \neq A \subset N_k}$ *a family of nonempty contractible subsets of* X_k *such that* $A \subset B$ *implies* $\Gamma_A^k \subset \Gamma_B^k$*. If* C_x^k*,* $x \in N_k$*,* $k \in I$*, are closed subsets of* $X = \prod_{k=1}^m X_k$ *such that for each* $k \in I$ *and* $A_k \in 2^{N_k}$, $\{\emptyset\}$,

$$\Gamma_{A_1}^1 \times \cdots \times \Gamma_{A_m}^m \subset \bigcap_{k=1}^m \bigcup_{x \in A_k} C_x^k,$$

then

$$\bigcap_{k=1}^m \bigcap_{x \in N_k} C_x^k \neq \emptyset.$$

Furthermore, if the product space X *is compact or some finite intersection of* C_x^k *is compact, then*

$$\bigcap_{k=1}^m \bigcap_{x \in D_k} C_x^k \neq \emptyset.$$

Proof: For each $k \in I$ and for each nonempty set $A_k \subset N_k$, by Horvath's theorem ([5], Theorem 1) , there is a continuous function $f_{N_k} : \Delta^{|N_k|-1} \to \Gamma_{N_k}^k$ and $f_{A_k} = f_{N_k}|_{A_k}$ such that $f_{A_k} : \Delta^{|A_k|-1} \to \Gamma_{A_k}^k$. We can choose $q_{N_k}(\alpha^k) = \{f_{N_k}(\alpha^k)\}$ for all $\alpha^k \in \Delta^{|A_k|-1}$. Then, for each $k \in I$, $(X_k, N_k, \{q_{A_k}\})$ forms a pseudo space, and

$$q_{N_1}(\Delta^{|A_1|-1}) \times \cdots \times q_{N_m}(\Delta^{|A_m|-1}) \subset \Gamma_{A_1}^1 \times \cdots \times \Gamma_{A_m}^m \subset \bigcap_{k=1}^m \bigcup_{x \in A_k} C_x^k.$$

By Theorem 2.1, we know that

$$\bigcap_{k=1}^{m}\bigcap_{x\in N_k} C_x^k \neq \varnothing.$$

Furthermore, if the product space X is compact or there are $k\in I$ and $x\in N_k$ such that C_x^k is compact, then

$$\bigcap_{k=1}^{m}\bigcap_{x\in D_k} C_x^k \neq \varnothing. \qquad \blacksquare$$

We can use the same way as the proof of Corollary 2.2 to prove the following results which is discussed on G-convex space. We note that for a G-convex space (X,D,Γ) also can form a pseudo space $(X,D,\{q_A\})$ if for each nonempty subset A of D and the corresponding continuous function $f_A:\Delta^{|A|-1}\to\Gamma_A$ and we define $q_A(\alpha)=\{f_A(\alpha)\}$ for $\alpha\in\Delta^{|A|-1}$.

Theorem 2.3: *For $k\in I$, let (X_k,D_k,Γ^k) be a G-convex space, N_k any nonempty finite set of D_k. If C_x^k, $x\in D_k$, $k\in I$, are compactly closed subsets of $X=\prod_{k=1}^{m} X_k$ such that for each $k\in I$ and $A_k\in 2^{N_k}\setminus\{\varnothing\}$,*

$$\Gamma_{A_1}^1\times\cdots\times\Gamma_{A_m}^m\subset\bigcap_{k=1}^{m}\bigcup_{x\in A_k} C_x^k,$$

then

$$\bigcap_{k=1}^{m}\bigcap_{x\in N_k} C_x^k \neq \varnothing.$$

Furthermore, if the product space X is compact or some finite intersection of C_x^k is compact, then

$$\bigcap_{k=1}^{m}\bigcap_{x\in D_k} C_x^k \neq \varnothing.$$

3. Coincidence theorems

From Theorem 2.3, we have the following coincidence results which is slight generalized the results duce to Park ([11], Theorem 1):

Theorem 3.1: *For* $k \in I$ *, let* (X_k, D_k, Γ^k) *be G-convex space,* $G_k : D_k \to 2^Y$ *and* $F : X \to 2^Y$ *such that*

(1) *for each* $k \in I$ *and* $x \in D_k$ *,* $F^+ G_k(x)$ *is compactly closed; and*

(2) *for each nonempty finite set* $A = \prod_{k\in I} A_k \in \prod_{k\in I} (2^{D_k} , \{\emptyset\})$ *,*

$$\Gamma^1_{A_1} \times \cdots \times \Gamma^m_{A_m} \subset \bigcap_{k\in I} \bigcup_{x\in A_k} F^+ G_k(x).$$

If the product space X *is compact or some finite intersection of* $F^+ G_k(x)$ *is compact, then there is a* $w \in X$ *such that* $F(w) \subset \bigcap_{k\in I} \bigcap_{x\in D_k} G_k(x)$.

Proof: We can choose $C^k_x = F^+ G_k(x)$ for each $k \in I$ and $x \in D_k$.
Applying Theorem 2.3, we have

$$\bigcap_{k=1}^{m} \bigcap_{x\in D_k} F^+ G_k(x) = \bigcap_{k=1}^{m} \bigcap_{x\in D_k} C^k_x \neq \emptyset.$$

But $\bigcap_{k=1}^{m} \bigcap_{x\in D_k} F^+ G_k(x) = F^+ (\bigcap_{k=1}^{m} \bigcap_{x\in D_k} G_k(x))$, there is a $w \in X$ such that

$F(w) \subset \bigcap_{k\in I} \bigcap_{x\in D_k} G_k(x)$.

Next, we derive another coincidence theorem that different from Theorem 3.1:

Theorem 3.2: *For* $k \in I$ *, let* (X_k, D_k, Γ^k) *be a G-convex space, the two mappings* $G_k : D_k \to 2^Y$ *and* $F : X \to 2^Y$ *have compactly closed values such that the family* $\{G_k(x) : k \in I, x \in D_k\}$ *has intersection property with respect to* F *and*

(1) *for each* $k \in I$ *and* $x \in D_k$ *,* $F^- G_k(x)$ *is compactly closed; and*

(2) *for each nonempty finite set* $A = \prod_{k \in I} A_k \in \prod_{k \in I} (2^{D_k}\ ,\ \ \{\emptyset\})$,

$$\Gamma_{A_1}^1 \times \cdots \times \Gamma_{A_m}^m \subset \bigcap_{k \in I} \bigcup_{x \in A_k} F^- G_k(x).$$

If F *has compact values or some finite intersection of* $G_k(x)$ *is compact, then there is a* $w \in X$ *such that* $F(w) \bigcap (\bigcap_{k \in I} \bigcap_{x \in D_k} G_k(x)) \neq \emptyset$.

Proof: We can choose $C_x^k = F^- G_k(x)$ for each $k \in I$ and $x \in D_k$. Applying Theorem 2.3, we have

$$\bigcap_{k=1}^{m} \bigcap_{x \in A_k} F^- G_k(x) = \bigcap_{k=1}^{m} \bigcap_{x \in A_k} C_x^k \neq \emptyset.$$

But $\bigcap_{k=1}^m \bigcap_{x \in A_k} F^- G_k(x) = F^- (\bigcap_{k=1}^m \bigcap_{x \in A_k} G_k(x))$, there is a $w \in X$ such that $F(w) \bigcap (\bigcap_{k \in I} \bigcap_{x \in A_k} G_k(x)) \neq \emptyset$. If F has compact values or some finite intersection of $G_k(x)$ is compact, then there is a $w \in X$ such that

$$F(w) \bigcap (\bigcap_{k \in I} \bigcap_{x \in D_k} G_k(x)) \neq \emptyset. \qquad \blacksquare$$

Now, we shall discuss the coincidence theorems on pseudo space as follows:

Theorem 3.3: *For* $k \in I$, *let* $(X_k, D_k, \{q_{A_k}\})$ *be a pseudo convex space,* N_k *any nonempty finite set of* D_k, $G_k : D_k \to 2^Y$ *and* $F : X \to 2^Y$ *such that the following conditions hold:*

(1) *for each* $k \in I$ *and* $x \in D_k$, $F^+ G_k(x)$ *is compactly closed;*

(2) *for each nonempty finite set* $A = \prod_{k \in I} A_k \in \prod_{k \in I} (2^{D_k}\ ,\ \ \{\emptyset\})$,

$$q_1(\Delta^{|A_1|-1}) \times \cdots \times q_m(\Delta^{|A_m|-1}) \subset \bigcap_{k \in I} \bigcup_{x \in A_k} F^+ G_k(x);$$

and

(3) *the family* $\{F^{+}G_k(x): k\in I, x\in D_k\}$ *has intersection with respect to* q_N *, where* q_N *is defined as the same as the one in the proof of Theorem 2.1 .*

If the product space X *is compact or some finite intersection of* $F^{+}G_k(x)$ *is compact, then there is a* $w\in X$ *such that* $F(w)\subset\bigcap_{k\in I}\bigcap_{x\in D_k} G_k(x)$.

Proof: We can choose $C_x^k = F^{+}G_k(x)$ for each $k\in I$ and $x\in D_k$.

Applying Theorem 2.1, we have

$$\bigcap_{k=1}^{m}\bigcap_{x\in D_k} F^{+}G_k(x)=\bigcap_{k=1}^{m}\bigcap_{x\in D_k} C_x^k\neq\varnothing.$$

But $\bigcap_{k=1}^{m}\bigcap_{x\in D_k} F^{+}G_k(x)=F^{+}(\bigcap_{k=1}^{m}\bigcap_{x\in D_k} G_k(x))$, there is a $w\in X$ such that

$$F(w)\subset\bigcap_{k\in I}\bigcap_{x\in D_k} G_k(x)\,.$$ ■

Theorem 3.3 generalize the Theorem 3.1 to pseudo space. Next, we derive another coincidence theorem that different from Theorem 3.3:

Theorem 3.4: *For* $k\in I$ *, let* $(X_k, D_k, \{q_{A_k}\})$ *be a pseudo convex space,* N_k *any nonempty finite set of* D_k *,* $G_k: D_k\to 2^{Y}$ *and* $F: X\to 2^{Y}$ *such that the family* $\{G_k(x): k\in I, x\in D_k\}$ *has intersection property with respect to* F *and the family* $\{F^{-}G_k(x): k\in I, x\in D_k\}$ *has intersection property with respect to* q *, and the following conditions hold:*

(1) *for each* $k\in I$ *and* $x\in D_k$ *,* $F^{-}G_k(x)$ *and* $G_k(x)$ *are compactly closed; and*

(2) *for each nonempty finite set* $A=\prod_{k\in I} A_k\in\prod_{k\in I}(2^{D_k}\,,\ \{\varnothing\})$,

$$q_1(\Delta^{|A_1|-1})\times\cdots\times q_m(\Delta^{|A_m|-1})\subset\bigcap_{k\in I}\bigcup_{x\in A_k} F^{-}G_k(x).$$

Then there is a $w \in X$ *such that* $F(w)\bigcap(\bigcap_{k\in I}\bigcap_{x\in N_k} G_k(x)) \neq \emptyset$. *Furthermore, if the product space* X *is compact or some finite intersection of* $G_k(x)$ *is compact, then* $F(w)\bigcap(\bigcap_{k\in I}\bigcap_{x\in D_k} G_k(x)) \neq \emptyset$.

Proof: We only need to check the f.i.p. hold for the collection $\{G_k(x): k \in I, x \in N_k\}$. Indeed, we can choose $C_x^k = F^- G_k(x)$ for each $k \in I$ and $x \in D_k$. Applying Theorem 2.1, we have

$$\bigcap_{k=1}^{m}\bigcap_{x\in N_k} F^- G_k(x) = \bigcap_{k=1}^{m}\bigcap_{x\in N_k} C_x^k \neq \emptyset.$$

But $\bigcap_{k=1}^{m}\bigcap_{x\in N_k} F^- G_k(x) = F^-(\bigcap_{k=1}^{m}\bigcap_{x\in N_k} G_k(x))$, there is a $w \in X$ such that

$$F(w)\bigcap(\bigcap_{k\in I}\bigcap_{x\in N_k} G_k(x)) \neq \emptyset. \qquad \blacksquare$$

Theorem 3.4 generalize Theorem 3.2 to pseudo space. We note that if we take $Y = X$ and F is the identity mapping on X, then both Theorem 3.3 and Theorem 3.4 can be reduced to Theorem 2.1 and both Theorem 3.1 and Theorem 3.2 can be reduced to Theorem 2.3.

4. Fixed point Results

Next, we establish a Fan-Browder's type fixed point theorem on pseudo space:

Theorem 4.1: *For each* $k \in I$, *let* $(X_k, \{q_{A_k}\})$ *be a pseudo space and* $S_k, T_k : X \to 2^{X_k}$ *be set-valued mappings such that*

(1) *for each* $k \in I$ *and* $x_k \in X_k$, *the set* $S_k^{-1}(x_k)$ *is compactly open,*

(2) *for each* $k \in I$ *and* $x \in X$, *if* $S_k(x)$ *is nonempty, then* $T_k(x)$ *is* q*-convex relative to* $S_k(x)$, *that is, for any nonempty finite subset* C_k *of*

$S_k(x)$, we have $q_{A_k}(\Delta^{|C_k|-1}) \subset T_k(x)$,

(3) *for each* $x \in X$ *, there exists* $k = k(x) \in I$ *such that* $S_k(x) \neq \varnothing$ *; and*

(4) *there are* $k \in I$ *and* $x_k \in X_k$ *such that* X , $S_k^{-1}(x_k)$ *is compact.*

Then there exist $\overline{x} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_m) \in X$ *and* $\overline{k} \in I$ *such that* $\overline{x}_{\overline{k}} \in T_{\overline{k}}(\overline{x})$.

Proof: For each $k \in I$, let $F_k, G_k : X_k \to 2^X$ be the mappings defined by $G_k(x_k) = X$, $T_k^{-1}(x_k)$ and $F_k(x_k) = X$, $S_k^{-1}(x_k)$ for $x_k \in X_k$. Then the mapping F_k has compactly closed for each $k \in I$. From (3), we have $\bigcup_{k=1}^m \bigcup_{x_k \in X_k} S_k^{-1}(x_k) = X$. This implies that

$$\bigcap_{k=1}^{m} \bigcap_{x_k \in X_k} F_k(x_k) = X \ , \quad \bigcup_{k=1}^{m} \bigcup_{x_k \in X_k} S_k^{-1}(x_k) = \varnothing.$$

Hence, by Theorem 2.1 with $D_k = X_k$ for each $k \in I$, there exist nonempty finite subsets A_k of X_k , $k \in I$, such that

$$q_1(\Delta^{|A_1|-1}) \times q_2(\Delta^{|A_2|-1}) \times \cdots \times q_m(\Delta^{|A_m|-1}) \not\subset \bigcap_{k=1}^{m} \bigcup_{x \in A_k} F_k(x).$$

We can choose some point $\overline{x} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_m)$ with

$$\overline{x} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_m) \in q_1(\Delta^{|A_1|-1}) \times q_2(\Delta^{|A_2|-1}) \times \cdots \times q_m(\Delta^{|A_m|-1}) \ , \quad \bigcap_{k=1}^{m} \bigcup_{x \in A_k} F_k(x) \quad .$$

Then $\overline{x} \in X$ and for some $\overline{k} \in I$, $\overline{x} \notin F_{\overline{k}}(x_{\overline{k}})$ for all $x_{\overline{k}} \in A_{\overline{k}}$. That is, $A_{\overline{k}} \subset S_{\overline{k}}(\overline{x})$. From (2), $q_{\overline{k}}(\Delta^{|A_{\overline{k}}|-1}) \subset T_{\overline{k}}(\overline{x})$. This implies that, for this $\overline{k}$,

$$\overline{x}_{\overline{k}} \in T_{\overline{k}}(\overline{x}). \qquad \blacksquare$$

By using the results of Theorem 4.1, we can derive the following generalization of Corollary 8 in [8] which generalized the Fan’s result ([4], Lemma 4):

Theorem 4.2: *For each* $k \in I$, *let* $(X_k, \{q_{A_k}\})$ *be a pseudo space.* $X = \prod_{k \in I} X_k$. *Let* $A_k, B_k \subset X_k \times X$ *for* $k \in I$.

(1) *for each* $k \in I$ *and* $y_k \in X_k$, *the set* $\{x \in X : (y_k, x) \in B_k\}$ *is compactly closed in* X,

(2) *for any* $x = (x_1, x_2, \ldots, x_m) \in X$ *and for each* $k \in I$, $(x_k, x) \in A_k$,

(3) *for any* $x \in X$, *if the sets* $\{y_k \in X_k : (y_k, x) \notin B_k\}$, $k \in I$, *are nonempty, then the set* $\{y_k \in X_k : (y_k, x) \notin A_k\}$ *is* q*-convex relative to the set* $\{y_k \in X_k : (y_k, x) \notin B_k\}$; *and*

(4) *there are* $k \in I$ *and* $y_k \in X_k$ *such that the set* $\{x \in X : (y_k, x) \in B_k\}$ *is compact.*

Then there is an $\bar{x} \in X$ *such that* $X_k \times \{\bar{x}\} \subset B_k$ *for each* $k \in I$.

Proof: For each $k \in I$, we define $S_k, T_k : X \to 2^{X_k}$ by

$$S_k(x) = \{y_k \in X_k : (y_k, x) \notin B_k\},$$

and

$$T_k(x) = \{y_k \in X_k : (y_k, x) \notin A_k\}.$$

By (1), $S_k^{-1}(y_k)$ is compactly open in X_k for each $k \in I$ and $y_k \in X_k$. For each $k \in I$, by (3), if $S_k(x)$ is nonempty, then for each nonempty finite subset C_k of $S_k(x)$, we have $q_{A_k}(\Delta^{|C_k|-1}) \subset T_k(x)$. From (2), we know that $x_k \notin T_k(x)$ for each $x \in X$ and $k \in I$. Applying Theorem 4.1, there is an $\bar{x} \in X$ such that $S_k(\bar{x}) = \varnothing$ for all $k \in I$. This implies

$$(x_k, \bar{x}) \in B_k \text{ for all } k \in I \text{ and } x_k \in X_k,$$

and hence, there is an $\bar{x} \in X$ such that $X_k \times \{\bar{x}\} \subset B_k$ for all $k \in I$.

5. System of Variational Inequalities

Now, for the family of the functions $\{f_k : X_k \times X \to R\}_{k \in I}$, we can consider the system of variational inequalities (in short, SVI) which is to find $\bar{x} \in X$ such that for each $k \in I$, $f_k(x_k, \overline{x}) \geq 0$, for all $x_k \in X_k$.

The existence theorem of the system of variational inequalities was discussed by Ansari and Yao [1]. Now we want to establish the existence theorem of system of variational inequalities on pseudo space.

Theorem 5.1: *For each* $k \in I$, *let* $(X_k, \{q_{A_k}\})$ *be a pseudo space and let* f_k *and* g_k *be two real-valued functions defined on* $X \times X_k$. *Suppose that*

(1) *for each* $k \in I$ *and* $y_k \in X_k$, *the mapping* $x \to f_k(y_k, x)$ *is upper semi-continuous on any compact set of* X,

(2) *for any* $x = (x_1, x_2, ..., x_m) \in X$ *and each* $k \in I$, $g_k(x_k, x) \geq 0$,

(3) *for any* $x \in X$ *and each* $k \in I$, *if the the set* $\{y_k \in X_k : f_k(y_k, x) < 0\}$ *is nonempty, then the set* $\{y_k \in X_k : g_k(y_k, x) < 0\}$ *is* q*-convex relative to the set* $\{y_k \in X_k : f_k(y_k, x) < 0\}$*; and*

(4) *there are* $k \in K$ *and* $y_k \in X_k$ *such that the set* $\{x \in X : f_k(y_k, x) \geq 0\}$ *is compact.*

Then there is an $\overline{x} \in X$ *such that for all* $x_k \in X_k$, $f_k(x_k, \overline{x}) \geq 0$ *for each* $k \in I$.

Proof: For each $k \in I$, let $A_k = \{(x_k, x) \in X_k \times X : g_k(x_k, x) \geq 0\}$ and $B_k = \{(x_k, x) \in X_k \times X : f_k(x_k, x) \geq 0\}$. Then all the conditions of Theorem 4.2 hold, and then we can deduce the conclusion that there is an $\overline{x} \in X$ such that $X_k \times \{\overline{x}\} \subset B_k$ for each $k \in I$. That is, there is an $\overline{x} \in X$ such that for each $k \in I$, $f_k(x_k, \overline{x}) \geq 0$ for all $x_k \in X_k$. ■

Theorem 5.2: *For each* $k \in I$, *let* $(X_k, \{q_{A_k}\})$ *be a pseudo space and let* f_k *and* g_k *be two real-valued functions defined on* $X \times X_k$. *Suppose that*

(1) *for each* $k \in I$ *and* $y_k \in X_k$, *the mapping* $x \to f_k(y_k, x)$ *is upper semi-continuous on any compact set of* X,

(2) *for any* $x = (x_1, x_2, ..., x_m) \in X$ *and each* $k \in I$, $g_k(x_k, x) \geq 0$,

(3) *for each* $k \in I$, $f_k(y_k, x) < 0$ *implies* $g_k(y_k, x) < 0$ *for all* $(y_k, x) \in X_k \times X$,

(4) *for each* $k \in I$ *and* $x \in X$, *either (a) if the the set*

$\{y_k \in X_k : f_k(y_k, x) < 0\}$ *is nonempty, then the set* $\{y_k \in X_k : f_k(y_k, x) < 0\}$ *is* q*-convex set, or (b) if the the set* $\{y_k \in X_k : g_k(y_k, x) < 0\}$ *is nonempty, then the set* $\{y_k \in X_k : g_k(y_k, x) < 0\}$ *is* q*-convex set; and*

(5) *there are* $k \in K$ *and* $y_k \in X_k$ *such that the set* $\{x \in X : f_k(y_k, x) \geq 0\}$ *is compact.*

Then there is an $\overline{x} \in X$ *such that for each* $k \in I$, $f_k(x_k, \overline{x}) \geq 0$, *for all* $x_k \in X_k$.

Proof: The result will follow from Theorem 5.1 when we show that: if the the set $\{y_k \in X_k : f_k(y_k, x) < 0\}$ is nonempty, then for any nonempty finite subset C_k of $\{y_k \in X_k : f_k(y_k, x) < 0\}$, we have $q_{A_k}(\Delta^{|C_k|-1}) \subset \{y_k \in X_k : g_k(y_k, x) < 0\}$.

For each $x \in X$, let C_k be any nonempty finite subset of $\{y_k \in X_k : f_k(y_k, x) < 0\}$. First, we assume that the condition (a) of (4) holds. Then we have $q_{A_k}(\Delta^{|C_k|-1}) \subset \{y_k \in X_k : f_k(y_k, x) < 0\}$. Then for each $y \in q_{A_k}(\Delta^{|C_k|-1})$, $f_k(y, x) < 0$.

From (3), $g_k(y, x) < 0$, and hence $q_{A_k}(\Delta^{|C_k|-1}) \subset \{y_k \in X_k : g_k(y_k, x) < 0\}$.

Secondly, we assume that the condition (b) of (4) holds. For each $y \in C_k$, $f_k(y, x) < 0$, and then we have $g_k(y, x) < 0$ by (3). Hence C_k is a finite subset of $\{y_k \in X_k : g_k(y_k, x) < 0\}$ and we have $q_{A_k}(\Delta^{|C_k|-1}) \subset \{y_k \in X_k : g_k(y_k, x) < 0\}$. ■

Next, let E_k be a Hausdorff topological vector space with its topological dual E_k^*, X_k be a nonempty convex subset of E_k and A_k be a function defined on X into E_k^* for $k \in I$. The case $f_k(y_k, x) = \langle A_k(x), y_k - x_k \rangle$ where $\langle \cdot, \cdot \rangle$ denotes the pairing between E_k^* and E_k for each $k \in I$ was discussed by Pang [10] with applications in equilibrium problems. Bianchi [2] proved the existence of solutions of the system of variational inequalities by using the Fan-KKM Theorem. The existence theorem was also studied by Cohen and Chaplais [3], Zhu and Marcotte [12].

Corollary 5.3: *For each* $k \in I$, *let* A_k *be a function defined on* X *into* E_k^*. *Suppose that*

(1) *for each* $k \in I$, A_k *is upper semi-continuous on any compact set of* X,

(2) *there is a family* $\{g_k : k \in I\}$ *of real-valued functions defined on* $X \times X_k$ *such that*

(a) *for each* $k \in I$, $< A_k(x), y_k - x_k >< 0$ *implies* $g_k(y_k, x) < 0$ *for all* $(y_k, x) \in X_k \times X$,

(b) *for any* $x = (x_1, x_2, ..., x_m) \in X$, $x_k \in X_k$ *and each* $k \in I$, $g_k(x_k, x) \geq 0$*; and*

(3) *there are* $k \in I$ *and* $y_k \in X_k$ *such that the set* $\{x \in X :< A_k(x), y_k - x_k >\geq 0\}$ *is compact.*

Then there is an $\bar{x} \in X$ *such that for each* $k \in I$, $< A_k(\bar{x}), y_k - \bar{x}_k >\geq 0$ *for all* $y_k \in X_k$.

Proof: For each $k \in I$, we define $f_k(y_k, x) =< A_k(x), y_k - x_k >$ for all $(y_k, x) \in X_k \times X$. Then, by Theorem 5.2, there is an $\bar{x} \in X$ such that for each $k \in I$, $< A_k(\bar{x}), y_k - \bar{x}_k >\geq 0$ for all $y_k \in X_k$. ■

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廣義 Peleg 型 KKM 定理及其應用

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摘 要

在我們先前的文章[7]之中，我們在擬空間探討連續選擇定理。擬空間的有趣性質值得我們進一步來加以探討。本文將進一步地探討擬空間上介紹 Peleg 型 KKM 定理，並且建立一些定點的結果，將本結果應用於解聯立變分不等式。

關鍵詞：擬空間、 q-凸集合、 KKM 定理、 Peleg 定理、 上(下)反映射、定點定理、 聯立變分不等式